ON GROUND STATES OF ROZIKOV MODEL ON THE CAYLEY TREE

G. I. BOTIROV

Abstract. In this paper we consider a model on a Cayley tree which has a finite radius of interactions, the model was first considered by Rozikov. We describe a set of periodic ground states of the model.

The Cayley tree.

The Cayley tree $\Im^k$ of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that each vertex of which lies on $k+1$ edges. Let $\Im^k = (V, L, i)$, where V is the set of vertexes of $\Im^k$, L is the set of edges of $\Im^k$, and i is the incidence function associating to each edge $l \in L$ its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *nearest neighboring vertexes*, and we write $< x, y >$. A collection of the pairs $< x_0, x_1 >, < x_1, x_2 >, \ldots, < x_{d-1}, y >$ is called *a path* from x to y. The distance $d(x, y), x, y \in V$ is the length of the shortest path from x to y in V.

For the fixed $x^0 \in V$ we set $W_n = \{x \in V \mid d(x, x^0) = n\}$,

$$V_n = \{x \in V \mid d(x, x^0) \leq n\}, \quad L_n = \{l =< x, y >\in L \mid x, y \in V_n\}.$$

It is known (see e.g. [2]) that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of oreder $k \geq 1$ and the group G_k, of the free products of $k+1$ cyclic groups $\{e, a_i\}$, $i = 1, \ldots, k+1$ of the second order (i.e. $a_i^2 = e, a_i^{-1} = a_i$) with generators $a_1, a_2, \ldots, a_{k+1}$.

Configuration Space and the model

We consider models where the spin takes values in the set $\Phi = \{1, 2, \ldots, q\}, q \geq 2$. For $A \subseteq V$ a spin configuration σ_A on A is defined as a function $x \in A \to \sigma_A(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^V$. We denote $\Omega = \Omega_V$ and $\sigma = \sigma_V$. Also we define a *periodic configuration* as a configuration $\sigma \in \Omega$ which is invariant under a subgroup of shifts $F_k \subset G_k$ of finite index.

More precisely, a configuration $\sigma \in V$ is called F_k- periodic if $\sigma(yx) = \sigma(x)$ for any $x \in G_k$ and $y \in F_k$.

For a given periodic configuration the index of the subgroup is called the period of the configuration. A configuration that is invariant with respect to all shifts is called *translational–invariant*.

For $A \subset V$ let us define a generalized Kronecker symbol (see [6]) as the function $U(\sigma_A) : \Omega_A \to \{|A|-1, |A|-2, \dots, |A| - min\{|A|, |\Phi|\}\}$, by

$$U(\sigma_A) = |A| - |\sigma_A \cap \Phi|, \tag{1}$$

where as before $\Phi = \{1, 2, \dots, q\}$ and $|\sigma_A \cap \Phi|$ is the number of different values of $\sigma_A(x), x \in A$. For instance if σ_A is a constant configuration then $|\sigma_A \cap \Phi| = 1$.

Note that if $|A| = 2$, say, $A = \{x, y\}$, then $U(\{\sigma(x), \sigma(y)\}) = \delta_{\sigma(x)\sigma(y)}$,

$$\delta_{\sigma(x)\sigma(y)} = \begin{cases} 1, & \sigma(x) = \sigma(y), \\ 0, & \sigma(x) \neq \sigma(y). \end{cases}$$

Fix $r \in N$ and put $r' = [\frac{r+1}{2}]$, where $[a]$ is the integer part of a. Denote by M_r the set of all balls $b_r(x) = \{y \in V : d(x, y) \leq r'\}$ with radius r', i.e. $M_r = \{b_r(x) : \ x \in V\}$.

We consider the energy of the configuration $\sigma \in \Omega$ is given by the formal Hamiltonian

$$H(\sigma) = -J \sum_{b \in M_r} U(\sigma_b), \tag{2}$$

where $J \in R$. This Hamiltonian was first considered by Rozikov [6].

Ground states

The ground states for the model defined on Z^d can, for example, be found in [3], [7].

Definition 1. A configuration φ is called the ground states of relative Hamiltonian H, if

$$U(\varphi_b) = U^{min} = min\{U(\sigma_b) : \sigma_b \in \Omega_b\} \text{ for any } b \in M_r.$$

In [1], [5] the ground states of Ising and Potts models with competing interactions of radius $r = 2$ on the Cayley tree were described.

Let $GS(H)$ be the set of all ground states, and let $GS_p(H)$ be the set of all periodic ground states.

Theorem 1. a) If $J > 0$, then for all $r \geq 1$ and $k \geq 2$ the set $GS(H)$ consists only configurations $\{\sigma^{(i)}, i = 1, 2, \dots, s\}$, where $\sigma^{(i)} \equiv i, \forall x \in V$;

b) Let $r = 2,\ J < 0,\quad q \geq 2^m$ and $k \in \{2^{m-1} - 1, \dots, q - 2\},\ m = 3, 4, \dots$ then there exists a normal subgroup F of index 2^m, such, that any F − periodic configuration σ is a ground state for Hamiltonian H i.e. $\sigma \in GS_p(H)$.

Proof a) Easily follows from (1), (2) and Definition 1.

b) Since $J < 0$ to construct a ground state it is necessary to consider configurations σ with a condition, that $U(\sigma_b) = 0$ for all $b \in M$, i.e. on any ball $b \in M$ the configuration σ is such that $\sigma(x) \neq \sigma(y)$ if $x \neq y$. Therefore we will construct a normal subgroup F of index 2^m such, that any element of the set $S_1(e) = \{e, a_1, \dots, a_{k+1}\}$ is not equivalent (with respect to F) to each other element of the set. Since $k + 2 \leq q$ we get $k \leq q - 2$. Consider a normal subgroup F of index 2^m, such that $F = F_{A_1} \cap \dots \cap F_{A_m}$ where $F_{A_i} = \{x \in G_k : \sum\limits_{j \in A_i} \omega_j(x) - \text{even}\}$, and $\omega_x(a_i)$ is the number of letter a_i, in nondeductible word x, $A_i \subset \{1, \dots, k+1\}, i = 1, \dots, m$. Now we shall construct $A_i,\quad i = 1, \dots, m$, so that all elements of any ball $b \in M$ were from different classes of equivalency.

Let's consider all possible configurations $\alpha : \{1, 2, \dots, m\} \to \{\text{e, o}\}$ (where "e" designates "even" and "o" designates "odd"). Let's notice, that number of such configurations is equal to 2^m. From them choose half, i.e. 2^{m-1} configurations with following properties: or the number of letters "e" in a configuration is more than number of letters "o", or the number of letters "e" in a configuration is equal to number of letters "o" and among the last there are no configurations coinciding at replacement "e" on letters "o". Let's denote these 2^{m-1} configurations by

$$\alpha_0 = \{\text{e, e, e}, \dots, \text{e}\} = (\alpha_{01}, \alpha_{02}, \dots, \alpha_{0m})$$

$$\alpha_1 = \{\text{o, e, e}, \dots, \text{e}\} = (\alpha_{11}, \alpha_{12}, \dots, \alpha_{1m})$$

$$\alpha_2 = \{\text{e, o, e}, \dots, \text{e}\} = (\alpha_{21}, \alpha_{22}, \dots, \alpha_{2m})$$

$$\alpha_3 = \{\text{e, e, o}, \dots, \text{e}\} = (\alpha_{31}, \alpha_{32}, \dots, \alpha_{3m})$$

$$\dots\ \dots$$

$$\alpha_{2^{m-1}} = \{\text{o, e, e}, \dots, \text{o}\} = (\alpha_{2^{m-1}1}, \alpha_{2^{m-1}2}, \dots, \alpha_{2^{m-1}m}).$$

We can define sets $A_i,\ i = 1, 2, \dots, m$, as follows

$$A_i = \{j \in \{1, 2, \dots, k\} :\ \alpha_{ji} - \text{odd}\} \cup \{k + 1\},\ i = 1, 2, \dots, m. \tag{3}$$

Let's notice, that A_i, $i = 1, 2, \dots m$, make sense if $k+1 \geq 2^{m-1}$ i.e. $k \geq 2^{m-1} - 1$. Check, that $F = F_{A_1} \cap \dots \cap F_{A_m}$, constructed by sets (3), satisfies conditions of the theorem. At first we shall prove, that $S_1(e)$ with respect to F divides into different non-equivalent elements: Denote $S_1(x) = \{y \in V : d(x, y) = 1\} = \{x, xa_1, \dots, xa_{k+1}\}, \gamma_i(x) = |S_1(x) \cap F_i|$. It is enough to prove, that $\gamma_i(x) = 0$ or 1 for any $x \in V$ and $i = 1, \dots, m$. By our construction one has $\gamma_i(e) \in \{0, 1\}$ for any $i = 1, \dots, m$. Hence, elements of the set $S_1(e)$ are not equivalent to each others, also they are not equivalent to e. Then by Theorem 3 of [4] elements of the set $S_1(x)$ are not equivalent to each others. By Theorem 1 of [4] we get $x \sim xa_i$ (i.e. x and xa_i belong to one class) if and only if $e \sim a_i$. By our construction $e \nsim a_i, \forall i = 1, \dots, k+1$ hence $x \nsim xa_i$; therefore, $\gamma_i(x) = 0$ or 1.

The theorem is proved.

Theorem 2. Let $r = 2$. a) if $J > 0$, then $|GS_p(H)| = q$;

b) If $J < 0$, then $|GS_p(H)| = C_q^{k+2}(k+2)!$

Proof. Case a) is trivial. In case b) for a given configuration φ_b, for which the energy $U(\varphi_b)$ is minimal, we can use Theorem 1 to construct the periodic configurations σ with period 2^m. In each case, the exact number of such ground states coincides with the number of different configurations σ_b, such that the energy $U(\sigma_b)$ is minimal for any $b \in M$. The theorem is proved.

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